ON THE CLOSURE OF THE EXTENDED BICYCLIC SEMIGROUP

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ABSTRACT. In the paper we study the semigroup $\mathscr{C}_{\mathbb{Z}}$ which is a generalization of the bicyclic semigroup. We describe main algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and prove that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also we show that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T. We show that the non-empty remainder of $\mathscr{C}_{\mathbb{Z}}$ in a topological inverse semigroup T consists of a group of units $H(1_T)$ of T and a two-sided ideal I of T in the case when $H(1_T)\neq\varnothing$ and $I\neq\varnothing$. In the case when T is a locally compact topological inverse semigroup and $I\neq\varnothing$ we prove that an ideal I is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $\mathscr{C}_{\mathbb{Z}}\cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup T is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

1. INTRODUCTION AND PRELIMINARIES

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [6, 7, 9, 10]. If Y is a subspace of a topological space X and $A\subseteq Y$, then by $\operatorname{cl}_Y(A)$ we shall denote the topological closure of A in Y. We denote by $\mathbb{N}$ the set of positive integers.

An algebraic semigroup S is called *inverse* if for any element $x\in S$ there exists the unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruence on S, and *group* if the quotient semigroup $S/\mathfrak{C}$ is a group.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e\leqslant f$ if and only if $ef=fe=e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order.

Let E be a semilattice and $e\in E$. We denote ${\downarrow}e=\{f\in E\mid f\leqslant e\}$ and ${\uparrow}e=\{f\in E\mid e\leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [7]):

$$a\mathscr{R}b \text{ if and only if } aS^1=bS^1;$$
$$a\mathscr{L}b \text{ if and only if } S^1a=S^1b;$$
$$a\mathscr{J}b \text{ if and only if } S^1aS^1=S^1bS^1;$$
$$\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L};$$
$$\mathscr{H}=\mathscr{L}\cap\mathscr{R}.$$

A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

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A *semitopological* (resp. *topological*) *semigroup* is a Hausdorff topological space together with a separately (resp. jointly) continuous semigroup operation [6, 18]. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*. A topology τ on a (inverse) semigroup S which turns S to be a topological (inverse) semigroup is called a (*inverse*) *semigroup topology* on S.

An element s of a topological semigroup S is called *topologically periodic* if for every open neighbourhood $U(s)$ of s in S there exists a positive integer $n \geqslant 2$ such that $s^n \in U(s)$. Obviously, if there exists a subgroup $H(e)$ with a neutral element e in S, then $s \in H(e)$ is topologically periodic if and only if for every open neighbourhood $U(e)$ of e in S there exists a positive integer n such that $s^n \in U(e)$.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array:

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [7, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen's result [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology and a topological semigroup S can contain the bicyclic semigroup $\mathscr{C}(p,q)$ as a dense subsemigroup only as an open subset [8]. Also Bertman and West in [5] proved that the bicyclic semigroup as a Hausdorff semitopological semigroup admits only the discrete topology. The problem of an embedding of the bicycle semigroup into compact-like topological semigroups solved in the papers [2, 3, 4, 11, 13] and the closure of the bicycle semigroup in topological semigroups studied in [8].

Let $\mathbb{Z}$ be the additive group of integers. On the Cartesian product $\mathscr{C}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b)\cdot(c,d) = \begin{cases} (a-b+c,d), & \text{if } b<c; \\ (a,d), & \text{if } b=c; \\ (a,d+b-c), & \text{if } b>c, \end{cases} \tag{1}$$

for $a,b,c,d \in \mathbb{Z}$. The set $\mathscr{C}_{\mathbb{Z}}$ with such defined operation is called the *extended bicycle semigroup* [19].

In this paper we study the semigroup $\mathscr{C}_{\mathbb{Z}}$. We describe main algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and prove that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also we show that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology. Next we study the closure $\operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T. We show that the non-empty remainder of $\mathscr{C}_{\mathbb{Z}}$ in a topological inverse semigroup T consists of a group of units $H(1_T)$ of T and a two-sided ideal I of T in the case when $H(1_T) \neq \varnothing$ and $I \neq \varnothing$. In the case when T is a locally compact topological inverse semigroup and $I \neq \varnothing$ we prove that an ideal I is topologically isomorphic to the discrete additive group of integers and describe the topology on the subsemigroup $\mathscr{C}_{\mathbb{Z}} \cup I$. Also we show that if the group of units $H(1_T)$ of the semigroup T is non-empty, then $H(1_T)$ is either singleton or $H(1_T)$ is topologically isomorphic to the discrete additive group of integers.

2. Algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}$

Proposition 2.1. *The following statements hold:*

(*i*) $E(\mathscr{C}_{\mathbb{Z}}) = \{(a,a) \mid a \in \mathbb{Z}\}$, *and* $(a,a) \leqslant (b,b)$ *in* $E(\mathscr{C}_{\mathbb{Z}})$ *if and only if* $a \geqslant b$ *in* $\mathbb{Z}$, *and hence* $E(\mathscr{C}_{\mathbb{Z}})$ *is isomorphic to the linearly ordered semilattice* $(\mathbb{Z}, \max)$;

(*ii*) $\mathscr{C}_{\mathbb{Z}}$ *is an inverse semigroup, and the elements* (a,b) *and* (b,a) *are inverse in* $\mathscr{C}_{\mathbb{Z}}$;

(*iii*) *for any idempotents* $e, f \in \mathscr{C}_{\mathbb{Z}}$ *there exists* $x \in \mathscr{C}_{\mathbb{Z}}$ *such that* $x \cdot x^{-1} = e$ *and* $x^{-1} \cdot x = f$;

(*iv*) *elements* (a,b) *and* (c,d) *of the semigroup* $\mathscr{C}_{\mathbb{Z}}$ *are:*

  (*a*) $\mathscr{R}$*-equivalent if and only if* $a = c$;

  (*b*) $\mathscr{L}$*-equivalent if and only if* $b = d$;

  (*c*) $\mathscr{H}$*-equivalent if and only if* $a = c$ *and* $b = d$;

  (*d*) $\mathscr{D}$*-equivalent for all* $a, b, c, d \in \mathbb{Z}$;

  (*e*) $\mathscr{J}$*-equivalent for all* $a, b, c, d \in \mathbb{Z}$;

(*v*) $\mathscr{C}_{\mathbb{Z}}$ *is a bisimple semigroup and hence it is simple;*

(*vi*) *if* $(a,b) \cdot (c,d) = (x,y)$ *in* $\mathscr{C}_{\mathbb{Z}}$ *then* $x - y = a - b + c - d$.

(*vii*) *every maximal subgroup of* $\mathscr{C}_{\mathbb{Z}}$ *is trivial.*

(*viii*) *for every integer* n *the subsemigroup* $\mathscr{C}_{\mathbb{Z}}[n] = \{(a,b) \mid a \geqslant n \;\&\; b \geqslant n\}$ *of* $\mathscr{C}_{\mathbb{Z}}$ *is isomorphic to the bicyclic semigroup* $\mathscr{C}(p,q)$, *and moreover an isomorphism* $h \colon \mathscr{C}_{\mathbb{Z}}[n] \to \mathscr{C}(p,q)$ *is defined by the formula* $((a,b))\,h = q^{a-n}p^{b-n}$;

(*ix*) $\mathscr{L}\mathscr{I}_{\mathscr{C}_{\mathbb{Z}}} = \{\mathscr{L}^a \mid a \in \mathbb{Z}\}$, *where* $\mathscr{L}^a = \{(x,y) \in \mathscr{C}_{\mathbb{Z}} \mid y \geqslant a\}$, *is the family of all left ideals of the semigroup* $\mathscr{C}_{\mathbb{Z}}$;

(*x*) $\mathscr{R}\mathscr{I}_{\mathscr{C}_{\mathbb{Z}}} = \{\mathscr{R}^a \mid a \in \mathbb{Z}\}$, *where* $\mathscr{R}^a = \{(x,y) \in \mathscr{C}_{\mathbb{Z}} \mid x \geqslant a\}$, *is the family of all right ideals of the semigroup* $\mathscr{C}_{\mathbb{Z}}$.

Proof. The proofs of statements (*i*), (*ii*), (*iii*), (*iv*), (*vi*), (*vii*) and (*viii*) are trivial. Statement (*v*) follows from statement (*iii*) and Lemma 1.1 of [16].

Simple verifications (see: formula (1)) show that

$$(a,b)\mathscr{C}_{\mathbb{Z}} = \{(x,y) \in \mathscr{C}_{\mathbb{Z}} \mid x \geqslant a\} \qquad \text{and} \qquad \mathscr{C}_{\mathbb{Z}}(a,b) = \{(x,y) \in \mathscr{C}_{\mathbb{Z}} \mid y \geqslant b\}$$

for every $(a,b) \in \mathscr{C}_{\mathbb{Z}}$. This completes the proof of statements (*ix*) and (*x*). □

Proposition 2.2. *Every non-trivial congruence* $\mathfrak{C}$ *on the semigroup* $\mathscr{C}_{\mathbb{Z}}$ *is a group congruence, and moreover the quotient semigroup* $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ *is isomorphic to a cyclic group.*

Proof. First we shall show that if two distinct idempotents (a,a) and (b,b) of $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent then the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is a group. Without loss of generality we can assume that $(a,a) \leqslant (b,b)$, i.e., $a \geqslant b$ in $\mathbb{Z}$. Then we have that

$$\begin{aligned}
(a,b) \cdot (b,b) \cdot (b,a) &= (a,a);\\
(a,b) \cdot (a,a) \cdot (b,a) &= (a+(a-b), a+(a-b));\\
(a,b) \cdot (a+(a-b), a+(a-b)) \cdot (b,a) &= (a+2(a-b), a+2(a-b));\\
\cdots \quad \cdots \quad \cdots \quad & \quad \cdots \quad \cdots\\
(a,b) \cdot (a+j(a-b), a+j(a-b)) \cdot (b,a) &= (a+(j+1)(a-b), a+(j+1)(a-b));\\
\cdots \quad \cdots \quad \cdots \quad & \quad \cdots \quad \cdots
\end{aligned}$$

This implies that for every non-negative integers i and j we have that

$$(a+i(a-b), a+i(a-b))\, \mathfrak{C}\, (a+j(a-b), a+j(a-b)).$$

If $b \geqslant k$ in $\mathbb{Z}$ for some integer k, then by Proposition 2.1(*viii*) we get that any two distinct idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[k]$ of $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent and hence Proposition 2.1(*viii*) and Corollary 1.32 from [7] imply that for every integer n all idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[n]$ are $\mathfrak{C}$-equivalent. This implies that all idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}[n]$ are $\mathfrak{C}$-equivalent.

Since the semigroup $\mathscr{C}_{\mathbb{Z}}$ is inverse we conclude that the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ contains only one idempotent and hence by Lemma II.1.10 from [17] the semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is a group.

Suppose that two distinct elements (a,b) and (c,d) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent. Since $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup, Lemma III.1.1 from [17] implies that $(a,a)\mathfrak{C}(c,c)$ and $(b,b)\mathfrak{C}(d,d)$. Since $(a,b)\neq(c,d)$ we have that either $(a,a)\neq(c,c)$ or $(b,b)\neq(d,d)$, and hence by the first part of the proof we get that all idempotents of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are $\mathfrak{C}$-equivalent.

Next we shall show that if $\mathfrak{C}_{mg}$ be a least group congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$, then the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}_{mg}$ is isomorphic to the additive group of integers $\mathbb{Z}$.

By Proposition 2.1(i) and Lemma III.5.2 from [17] we have that elements (a,b) and (c,d) are $\mathfrak{C}_{mg}$-equivalent in $\mathscr{C}_{\mathbb{Z}}$ if and only if there exists an integer n such that $(a,b)\cdot(n,n)=(c,d)\cdot(n,n)$. Then Proposition 2.1$(i)$ implies that $(a,b)\cdot(g,g)=(c,d)\cdot(g,g)$ for any integer g such that $g\geqslant n$ in $\mathbb{Z}$. If $g\geqslant b$ and $g\geqslant d$ in $\mathbb{Z}$, then the semigroup operation in $\mathscr{C}_{\mathbb{Z}}$ implies that $(a,b)\cdot(g,g)=(g-b+a,g)$ and $(c,d)\cdot(g,g)=(g-d+c,g)$, and since $\mathbb{Z}$ is the additive group of integers we get that $a-b=c-d$. Converse, suppose that (a,b) and (c,d) are elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $a-b=c-d$. Then for any element $g\in\mathbb{Z}$ such that $g\geqslant b$ and $g\geqslant d$ in $\mathbb{Z}$ we have that $(a,b)\cdot(g,g)=(g-b+a,g)$ and $(c,d)\cdot(g,g)=(g-d+c,g)$, and since $a-b=c-d$ we get that $(a,b)\mathfrak{C}_{mg}(c,d)$. Therefore, $(a,b)\mathfrak{C}_{mg}(c,d)$ in $\mathscr{C}_{\mathbb{Z}}$ if and only if $a-b=c-d$.

We determine a map $\mathfrak{f}\colon\mathscr{C}_{\mathbb{Z}}\to\mathbb{Z}$ by the formula $((a,b))\mathfrak{f}=a-b$, for $a,b\in\mathbb{Z}$. Proposition 2.1(vi) implies that such defined map $\mathfrak{f}\colon\mathscr{C}_{\mathbb{Z}}\to\mathbb{Z}$ is a homomorphism. Then we have that $(a,b)\mathfrak{C}_{mg}(c,d)$ if and only if $((a,b))\mathfrak{f}=((c,d))\mathfrak{f}$, for $(a,b),(c,d)\in\mathscr{C}_{\mathbb{Z}}$, and hence the homomorphism $\mathfrak{f}$ generates the least group congruence $\mathfrak{C}_{mg}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$.

If $\mathfrak{c}$ is any congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ then the mapping $\mathfrak{c}\mapsto\mathfrak{c}\vee\mathfrak{C}_{mg}$ maps the congruence $\mathfrak{c}$ onto a group congruence $\mathfrak{c}\vee\mathfrak{C}_{mg}$, where $\mathfrak{C}_{mg}$ is the least group congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ (cf. [17, Section III]). Therefore every homomorphic image of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a homomorphic image of the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$, i.e., it is a homomorphic image of the additive group of integers $\mathbb{Z}$. This completes the proof of the theorem. □

3. The semigroup $\mathscr{C}_{\mathbb{Z}}$: topologizations and closures of $\mathscr{C}_{\mathbb{Z}}$ in topological semigroups

Theorem 3.1. *Every Hausdorff topology τ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $(\mathscr{C}_{\mathbb{Z}},\tau)$ is a semitopological semigroup is discrete, and hence $\mathscr{C}_{\mathbb{Z}}$ is a discrete subspace of any semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup.*

Proof. We fix an arbitrary idempotent (a,a) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and suppose that (a,a) is a non-isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Since the maps $\lambda_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ and $\rho_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ defined by the formulae $((x,y))\,\lambda_{(a,a)}=(a,a)\cdot(x,y)$ and $((x,y))\,\rho_{(a,a)}=(x,y)\cdot(a,a)$ are continuous retractions we conclude that $(a,a)\mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}(a,a)$ are closed subsets in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. We put

$$\mathsf{DL}_{(a,a)}\left[(a,a)\right]=\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}\mid (x,y)\cdot(a,a)=(a,a)\right\}.$$

Simple verifications show that

$$\mathsf{DL}_{(a,a)}\left[(a,a)\right]=\left\{(x,x)\in\mathscr{C}_{\mathbb{Z}}\mid x\leqslant a \text{ in } \mathbb{Z}\right\},$$

and since right translations are continuous maps in $(\mathscr{C}_{\mathbb{Z}},\tau)$ we get that $\mathsf{DL}_{(a,a)}\left[(a,a)\right]$ is a closed subset of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Then there exists an open neighbourhood $W_{(a,a)}$ of the point (a,a) in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ such that

$$W_{(a,a)}\subseteq\mathscr{C}_{\mathbb{Z}}\setminus\big((a+1,a+1)\mathscr{C}_{\mathbb{Z}}\cup\mathscr{C}_{\mathbb{Z}}(a+1,a+1)\cup\mathsf{DL}_{(a-1,a-1)}(a-1,a-1)\big).$$

Since $(\mathscr{C}_{\mathbb{Z}},\tau)$ is a semitopological semigroup we conclude that there exists an open neighbourhood $V_{(a,a)}$ of the idempotent (a,a) in the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ such that the following conditions hold:

$$V_{(a,a)}\subseteq W_{(a,a)},\qquad (a,a)\cdot V_{(a,a)}\subseteq W_{(a,a)}\qquad\text{and}\qquad V_{(a,a)}\cdot(a,a)\subseteq W_{(a,a)}.$$

Hence at least one of the following conditions holds:

(a) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in\mathscr{C}_{\mathbb{Z}}$ such that $x<y\leqslant a$; or

(b) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in\mathscr{C}_{\mathbb{Z}}$ such that $y<x\leqslant a$.

In case (a) we have that

$$(a,a)\cdot(x,y)=\big(a,a+(y-x)\big)\notin W_{(a,a)},$$

because $y-x\geqslant 1$, and in case (b) we have that

$$(x,y)\cdot(a,a)=\big(a+(x-y),a\big)\notin W_{(a,a)},$$

because $x-y\geqslant 1$, a contradiction. The obtained contradiction implies that the set $V_{(a,a)}$ is singleton, and hence the idempotent (a,a) is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$.

Let (a,b) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and suppose that (a,b) is a non-isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Since all right translations are continuous maps in $(\mathscr{C}_{\mathbb{Z}},\tau)$ and every idempotent (a,a) of $\mathscr{C}_{\mathbb{Z}}$ is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$ we conclude that

$$\mathsf{DL}_{(b,a)}\left[(a,a)\right]=\big\{(x,y)\in\mathscr{C}_{\mathbb{Z}}\mid (x,y)\cdot(b,a)=(a,a)\big\}$$

is a closed-and-open subset of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. Simple verifications show that

$$\mathsf{DL}_{(b,a)}\left[(a,a)\right]=\big\{(x,y)\in\mathscr{C}_{\mathbb{Z}}\mid x-y=a-b \;\text{ and }\; x\leqslant a\big\}.$$

Then we have that

$$\big\{(a,b)\big\}=\mathsf{DL}_{(b,a)}\left[(a,a)\right]\setminus\mathsf{DL}_{(b-1,a-1)}\left[(a-1,a-1)\right],$$

and hence (a,b) is an isolated point of the topological space $(\mathscr{C}_{\mathbb{Z}},\tau)$. This completes the proof of the theorem. □

Theorem 3.1 implies the following:

Corollary 3.2. *Every Hausdorff semigroup topology τ on $\mathscr{C}_{\mathbb{Z}}$ is discrete, and hence $\mathscr{C}_{\mathbb{Z}}$ is a discrete subspace of any topological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup.*

Since every discrete topological space is locally compact, Theorem 3.1 and Theorem 3.3.9 from [9] imply the following:

Corollary 3.3. *Let T be a semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a subsemigroup. Then $\mathscr{C}_{\mathbb{Z}}$ is an open subsemigroup of T.*

Lemma 3.4. *Let T be a Hausdorff semitopological semigroup which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Let $f\in T\setminus\mathscr{C}_{\mathbb{Z}}$ be an idempotent of the semigroup T which satisfies the property: there exists an idempotent $(n,n)\in\mathscr{C}_{\mathbb{Z}}$, $n\in\mathbb{Z}$, such that $(n,n)\leqslant f$. Then the following statements hold:*

(i) there exists an open neighbourhood $U(f)$ of f in T such that $U(f)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$;

(ii) f is the unit of T.

Proof. (i) Let $W(f)$ be an arbitrary open neighbourhood of the idempotent f in T. We fix an arbitrary element $(n,n)\in\mathscr{C}_{\mathbb{Z}}$, $n\in\mathbb{Z}$. By Corollary 3.3 the element (n,n) is an isolated point in T, and since T is a semitopological semigroup we have that there exists an open neighbourhood $U(f)$ of f in T such that

$$U(f)\subseteq W(f),\qquad U(f)\cdot\{(n,n)\}=\{(n,n)\}\qquad\text{and}\qquad \{(n,n)\}\cdot U(f)=\{(n,n)\}.$$

If the set $U(f)$ contains a non-idempotent element $(x,y)\in\mathscr{C}_{\mathbb{Z}}$, then Proposition 2.1($vi$) implies that $(x,y)\cdot(n,n),(n,n)\cdot(x,y)\notin E(\mathscr{C}_{\mathbb{Z}})$, a contradiction. The obtained contradiction implies the statement of the assertion.

(ii) First we show that $f\cdot(k,l)=(k,l)\cdot f=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$.

Suppose the contrary: there exists an element $(k,l)\in\mathscr{C}_{\mathbb{Z}}$ such that $x=f\cdot(k,l)\neq(k,l)$ for some $x\in T$. Let $U(x)$ be an open neighbourhood of x in T such that $(k,l)\notin U(x)$. Since T is a semitopological semigroup we get that there exists an open neighbourhood $V(f)$ of f in T such that $V(f)\cdot\{(k,l)\}\subseteq U(x)$. Again, since for an arbitrary integer a the maps $\lambda_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ and $\rho_{(a,a)}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ defined by the formulae $((x,y))\,\lambda_{(a,a)}=(a,a)\cdot(x,y)$ and $((x,y))\,\rho_{(a,a)}=(x,y)\cdot$

(a,a) are continuous retractions we conclude that statement (i) implies that there exists an open neighbourhood $W(f)$ of f in T such that $W(f)\subseteq V(f)$, $W(f)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$ and the following condition holds:

$$(p,p)\in W(f)\cap\mathscr{C}_{\mathbb{Z}} \qquad \text{if and only if} \qquad p\geqslant k.$$

Then $(p,p)\cdot(k,l)=(k,l)\notin U(x)$ for every $(p,p)\in W(f)\cap\mathscr{C}_{\mathbb{Z}}$, a contradiction. The obtained contradiction implies that $f\cdot(k,l)=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$. Similar arguments show that $(k,l)\cdot f=(k,l)$ for every $(k,l)\in\mathscr{C}_{\mathbb{Z}}$.

Next we show that $f\cdot x=x\cdot f=x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: there exists an element $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ such that $y=f\cdot x\neq x$ for some $y\in T$. Let $U(x)$ and $U(y)$ be open neighbourhoods of x and y in T, respectively, such that $U(x)\cap U(y)=\varnothing$. Since T is a semitopological semigroup we get that there exists an open neighbourhood $V(x)$ of x in T such that $V(x)\subseteq U(x)$ and $f\cdot V(x)\subseteq U(y)$. Again, since $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ we have that the set $V(x)\cap\mathscr{C}_{\mathbb{Z}}$ is infinite, and the previous part of the proof of the statement implies that $f\cdot(V(x)\cap\mathscr{C}_{\mathbb{Z}})\subseteq(V(x)\cap\mathscr{C}_{\mathbb{Z}})$. But we have that $V(x)\cap U(y)=\varnothing$, a contradiction. The obtained contradiction implies the equality $f\cdot x=x$. Similar arguments show that $x\cdot f=x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. □

Remark 3.5. We observe that the assertion (i) of Lemma 3.4 holds for right-topological and left-topological monoids.

Lemma 3.6. *Let T be a Hausdorff topological monoid with the unit 1_T which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Then the following assertions hold:*

(i) there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$;

and if the group of units $H(1_T)$ of T is non-singleton, then:

(ii) for every $x\in H(1_T)$ there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$;

(iii) for distinct $x,y\in H(1_T)$ there exist open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively, such that $a-b\neq c-d$ for every $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and for every $(c,d)\in U(y)\cap\mathscr{C}_{\mathbb{Z}}$;

(iv) the group $H(1_T)$ is torsion free;

(v) the group of units $H(1_T)$ of T is a discrete subgroup in T;

(vi) the group of units $H(1_T)$ of T is isomorphic to the infinite cyclic group;

(vii) every non-identity element of the group of units $H(1_T)$ in the semigroup T is not topologically periodic.

Proof. Statement (i) follows from Lemma 3.4(i).

(ii) In the case $H(1_T)=\{1_T\}$ statement (i) implies our assertion. Hence we suppose that $H(1_T)\neq\{1_T\}$ and let $x\in H(1_T)\setminus\{1_T\}$. By statement (i) there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$. Then the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(x)$ and $U(x^{-1})$ in the topological space T of x and the inverse element x^{-1} of x in $H(1_T)$, respectively, such that

$$U(x)\cdot U(x^{-1})\subseteq U(1_T) \qquad \text{and} \qquad U(x^{-1})\cdot U(x)\subseteq U(1_T).$$

Since $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$ we have that Proposition 2.1(vi) implies that $a-b+u-v=c-d+u-v$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and some $(u,v)\in U(x^{-1})\cap\mathscr{C}_{\mathbb{Z}}$, and hence $a-b=c-d$.

(iii) Suppose the contrary: there exist distinct $x,y\in H(1_T)$ and for all open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively, there are $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $(c,d)\in U(y)\cap\mathscr{C}_{\mathbb{Z}}$ such that $a-b=c-d$. The Hausdorffness of T implies that without loss of generality we can assume that $U(x)\cap U(y)=\varnothing$. Then statement (i) and the continuity of the semigroup operation in T imply that there exist open neighbourhoods $V(1_T)$, $V(x)$ and $V(y)$ of 1_T, x and y in T, respectively, such that

$$V(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}}),\ V(x)\subseteq U(x),\ V(y)\subseteq U(y),\ V(1_T)\cdot V(x)\subseteq U(x)$$
$$\text{and}\ \ V(1_T)\cdot V(y)\subseteq U(y).$$

Since by Theorem 1.7 from [6, Vol. 1] the sets $(a,a)T$ and $T(a,a)$ are closed in T for every idempotent $(a,a)\in\mathscr{C}_{\mathbb{Z}}$ and both neighbourhoods $V(x)$ and $V(y)$ contain infinitely many elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ we conclude that for every $(p,p)\in V(1_T)\cap\mathscr{C}_{\mathbb{Z}}$ there exist $(k,l)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $(m,n)\in V(y)\cap\mathscr{C}_{\mathbb{Z}}$ such that

$$p>k>m, \qquad p>l>n \qquad \text{and} \qquad k-l=m-n.$$

Then we get that

$$(p,p)\cdot(k,l)=(p,p+(l-k)) \qquad \text{and} \qquad (p,p)\cdot(m,n)=(p,p+(n-m)),$$

a contradiction. The obtained contradiction implies our assertion.

(iv) Suppose the contrary: there exist $x\in H(1_T)\setminus\{1_T\}$ and a positive integer n such that $x^n=1_T$. Then by statement (i) there exists an open neighbourhood $U(1_T)$ of the unit 1_T in T such that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$. The continuity of the semigroup operation in T and statement (ii) imply that there exists an open neighbourhood $V(x)$ of x in T such that $a-b=c-d$ for all $(a,b),(c,d)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$ and $\underbrace{V(x)\cdot\ldots\cdot V(x)}_{n\text{-times}}\subseteq U(1_T)$. We fix an arbitrary element $(a,b)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$. If $(a,b)^n=(x,y)$, then Proposition 2.1(vi) implies that $x-y=n\cdot(a-b)$ and since $x\neq 1_T$ we get that $(x,y)\notin U(1_T)$, a contradiction. The obtained contradiction implies statement (iv).

(v) Statement (iv) implies that the group of units $H(1_T)$ is infinite.

We fix an arbitrary $x\in H(1_T)$ and suppose that x is not an isolated point of $H(1_T)$. Then by statement (ii) there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Since the point x is not isolated in $H(1_T)$ we conclude that there exists $y\in H(1_T)$ such that $y\in U(x)$. Hence the set $U(x)$ is an open neighbourhood of y in T. Statement (iii) implies that there exist open neighbourhoods $W(x)\subseteq U(x)$ and $W(y)\subseteq U(x)$ of x and y in T, respectively, such that $a-b\neq c-d$ for every $(a,b)\in W(x)\cap\mathscr{C}_{\mathbb{Z}}$ and for every $(c,d)\in W(y)\cap\mathscr{C}_{\mathbb{Z}}$. This contradicts the choice of the neighbourhood $U(x)$. The obtained contradiction implies that every $x\in H(1_T)$ is an isolated point of $H(1_T)$.

(vi) Since the group of units $H(1_T)$ is not trivial, i.e., the group $H(1_T)$ is non-singleton, we fix an arbitrary $x\in H(1_T)\setminus\{1_T\}$. Then by statement (iv) we have that $x^n\neq 1_T$ for any positive integer n. Statement (ii) implies that there exists an open neighbourhood $U(x)$ in T such that $a-b=c-d$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. We define the map $\varphi\colon H(1_T)\to\mathbb{Z}$ by the following way: $(x)\varphi=k$ if and only if $a-b=k$ for every $(a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then statement (iv) and Proposition 2.1(vi) imply that the map $\varphi\colon H(1_T)\to\mathbb{Z}$ is an injective homomorphism. Obviously that $(H(1_T))\varphi$ is a subgroup in the additive group of integers. We fix the least positive integer $p\in(H(1_T))\varphi$. Then the element p generates the subgroup $(H(1_T))\varphi$ in the additive group of integers $\mathbb{Z}$, and hence the group $(H(1_T))\varphi$ is cyclic.

(vii) We fix an arbitrary element $x\in H(1_T)\setminus\{1_T\}$. Suppose the contrary: x is a topologically periodic element of S. Then there exist open neighbourhoods $U(1_T)$ and $U(x)$ of 1_T and x in T, respectively, such that $U(1_T)\cap U(x)=\varnothing$. Statements (i) and (iii) imply that without loss of generality we can assume that $U(1_T)\cap\mathscr{C}_{\mathbb{Z}}\subseteq E(\mathscr{C}_{\mathbb{Z}})$, and $a-b=c-d\neq 0$ for all $(a,b),(c,d)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then the topologically periodicity of x implies that there exists a positive integer n such that $x^n\in U(1_T)$. Since the semigroup operation in T is continuous we conclude that there exists an open neighbourhood $V(x)$ of x in T such that $\underbrace{V(x)\cdot\ldots\cdot V(x)}_{n\text{-times}}\subseteq U(1_T)$. We fix an arbitrary element $(a,b)\in V(x)\cap\mathscr{C}_{\mathbb{Z}}$. Then we have that $(a,b)^n\in U(1_T)\cap\mathscr{C}_{\mathbb{Z}}$ and hence $n(a-b)=0$, a contradiction. The obtained contradiction implies assertion (vii). □

Proposition 3.7. *Let G be non-trivial subgroup of the additive group of integers $\mathbb{Z}$ and $n\in\mathbb{Z}$. Then the subsemigroup H which is generated by the set $\{n\}\cup G$ is a cyclic subgroup of $\mathbb{Z}$.*

Proof. Without loss of generality we can assume that $n\in\mathbb{Z}\setminus G$ and $n>0$.

Since every subgroup of a cyclic group is cyclic (see [14, P. 47]), we have that G is a cyclic subgroup in $\mathbb{Z}$. We fix a generating element k of G such that $k>0$. Then we have that

$$(\underbrace{n+\cdots+n}_{(k-1)\text{-times}})-(\underbrace{k+\cdots+k}_{n\text{-times}})+n=0,$$

and hence we have that $-n\in H$. Since $\mathbb{Z}$ is a commutative group we conclude that H is a subgroup in $\mathbb{Z}$, which is generated by elements n and k, and hence H is a cyclic subgroup in $\mathbb{Z}$. $\square$

Proposition 3.8. *Let T be a Hausdorff topological monoid with the unit 1_T which contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup. Then the following assertions hold:*

(i) *if the set $L_{\mathscr{C}_{\mathbb{Z}}}=\{x\in T\setminus\mathscr{C}_{\mathbb{Z}}\mid$ there exists $y\in\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot y\in\mathscr{C}_{\mathbb{Z}}\}$ is non-empty, then $L_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T, and moreover if $a\in L_{\mathscr{C}_{\mathbb{Z}}}$, then there exists an open neighbourhood $U(a)$ of a in T such that $n_1-m_1=n_2-m_2$ for all $(n_1,m_1),(n_2,m_2)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$;*

(ii) *if the set $R_{\mathscr{C}_{\mathbb{Z}}}=\{x\in T\setminus\mathscr{C}_{\mathbb{Z}}\mid$ there exists $y\in\mathscr{C}_{\mathbb{Z}}$ such that $y\cdot x\in\mathscr{C}_{\mathbb{Z}}\}$ is non-empty, then $R_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T, and moreover if $a\in R_{\mathscr{C}_{\mathbb{Z}}}$, then there exists an open neighbourhood $U(a)$ of a in T such that $n_1-m_1=n_2-m_2$ for all $(n_1,m_1),(n_2,m_2)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$;*

(iii) *if the set $L_{\mathscr{C}_{\mathbb{Z}}}$ (resp., $R_{\mathscr{C}_{\mathbb{Z}}}$) is non-empty, then for every $a\in L_{\mathscr{C}_{\mathbb{Z}}}$ (resp., $a\in R_{\mathscr{C}_{\mathbb{Z}}}$) there exist an open neighbourhood $U(a)$ of a in T and an integer n_a such that $p\leqslant n_a$ and $q\leqslant n_a$ for all $(p,q)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$;*

(iv) $L_{\mathscr{C}_{\mathbb{Z}}}=R_{\mathscr{C}_{\mathbb{Z}}}$;

(v) *$\uparrow\mathscr{C}_{\mathbb{Z}}=\mathscr{C}_{\mathbb{Z}}\cup L_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T and $\mathscr{C}_{\mathbb{Z}}$ is a minimal ideal in $\uparrow\mathscr{C}_{\mathbb{Z}}$;*

(vi) *if for an element $a\in T\setminus\mathscr{C}_{\mathbb{Z}}$ there is an open neighbourhood $U(a)$ of a in T and the following conditions hold:*
  (a) *$m_1-m_2=n_1-n_2$ for all $(m_1,n_1),(m_2,n_2)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$; and*
  (b) *there exists an integer n_a such that $n\leqslant n_a$ and $m\leqslant n_a$ for every $(m,n)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$,*
  then $a\in L_{\mathscr{C}_{\mathbb{Z}}}$;

(vii) *if $I=T\setminus\uparrow\mathscr{C}_{\mathbb{Z}}\neq\varnothing$, then I is an ideal of T;*

(viii) *the set*

$$\begin{aligned}\uparrow(a,b)&=\{x\in T\mid x\cdot(b,b)=(a,b)\}\\&=\{x\in T\mid (a,a)\cdot x=(a,b)\}\\&=\{x\in T\mid (a,a)\cdot x\cdot(b,b)=(a,b)\}\end{aligned}$$

is closed-and-open in T for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}$;

(ix) *the set $\uparrow(a,b)\cap L_{\mathscr{C}_{\mathbb{Z}}}$ is either singleton or empty;*

(x) *$L_{\mathscr{C}_{\mathbb{Z}}}$ is isomorphic to a submonoid of the additive group of integers $\mathbb{Z}$, and moreover if a maximal subgroup of $L_{\mathscr{C}_{\mathbb{Z}}}$ is non-singleton, then $L_{\mathscr{C}_{\mathbb{Z}}}$ is isomorphic to the additive group of integers $\mathbb{Z}$;*

(xi) *$\uparrow\mathscr{C}_{\mathbb{Z}}$ is an open subset in T, and hence if $I=T\setminus\uparrow\mathscr{C}_{\mathbb{Z}}\neq\varnothing$, then the ideal I is a closed subset in T;*

(xii) *if the semigroup T contains a non-singleton group of units $H(1_T)$, then $H(1_T)=T\setminus(\mathscr{C}_{\mathbb{Z}}\cup I)$.*

Proof. (i) We observe that since $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup we conclude that $x\in L_{\mathscr{C}_{\mathbb{Z}}}$ if and only if there exists an idempotent $e\in\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot e\in\mathscr{C}_{\mathbb{Z}}$, for $x\in T$.

We fix an arbitrary $x\in L_{\mathscr{C}_{\mathbb{Z}}}$. Let (n,n) be an idempotent in $\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)=x\cdot(n,n)\in\mathscr{C}_{\mathbb{Z}}$. Then by Corollary 3.2 we have that (n,n) and (a,b) are isolated points in T, and the continuity of the semigroup operation in T implies that there exists an open neighbourhood $U(x)$ of x in T such that

$$U(x)\cdot\{(n,n)\}=\{(a,b)\}\in\mathscr{C}_{\mathbb{Z}}.$$

Then Proposition 2.1(vi) implies that $p-q=a-b$ for all $(p,q)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Also, since

$$(p,q)(n,n)=\begin{cases}(p-q+n,n), & \text{if } q\leqslant n;\\ (p,q), & \text{if } q\geqslant n\end{cases}\tag{2}$$

we have that $q \leqslant n = b$.

Suppose that $x, y \in L_{\mathscr{C}_{\mathbb{Z}}}$, and (i,i) and (j,j) are idempotents in $\mathscr{C}_{\mathbb{Z}}$ such that $x \cdot (i,i) = (k,l) \in \mathscr{C}_{\mathbb{Z}}$ and $y \cdot (j,j) \in \mathscr{C}_{\mathbb{Z}}$, $i,j,k,l \in \mathbb{Z}$. We fix an arbitrary integer d such that $d \geqslant \max\{k,j\}$. Then we have that

$$\begin{aligned}
(y \cdot x) \cdot ((i,i) \cdot (l,k) \cdot (d,d)) &= y \cdot (x \cdot (i,i) \cdot (l,k) \cdot (d,d)) \\
&= y \cdot ((k,l) \cdot (l,k) \cdot (d,d)) \\
&= y \cdot ((k,k) \cdot (d,d)) \\
&= y \cdot (d,d) \\
&= y \cdot ((j,j) \cdot (d,d)) \\
&= (y \cdot (j,j)) \cdot (d,d) \in \mathscr{C}_{\mathbb{Z}}.
\end{aligned}$$

This implies that $L_{\mathscr{C}_{\mathbb{Z}}}$ is a subsemigroup of T and completes the proof of our assertion.

The proof of assertion (ii) is similar to (i).

Statement (i) and formula (2) imply assertion (iii). In the case $a \in R_{\mathscr{C}_{\mathbb{Z}}}$ the proof is similar.

(iv) Let be $L_{\mathscr{C}_{\mathbb{Z}}} \neq \varnothing$. We fix an arbitrary element $a \in L_{\mathscr{C}_{\mathbb{Z}}}$. Then there exists an idempotent $(i_a, i_a) \in \mathscr{C}_{\mathbb{Z}}$ such that $a \cdot (i_a, i_a) = (i,j) \in \mathscr{C}_{\mathbb{Z}}$. Assertion (iii) implies that there exist an open neighbourhood $U(a)$ of a in T and an integer n_a such that $n - m = i - j$, $n \leqslant n_a$ and $m \leqslant n_a$ for all $(n,m) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$. Without loss of generality we can assume that $i_a \geqslant n_a$.

We shall show that $(i_a, i_a) \cdot a \in \mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: $(i_a, i_a) \cdot a = b \in T \setminus \mathscr{C}_{\mathbb{Z}}$. Assertion (iii) implies that there exist integers

$$n_0(a) = \max\{n \mid (n,m) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}\} \qquad \text{and} \qquad m_0(a) = \max\{m \mid (n,m) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}\}.$$

Since $i_a \geqslant n_a$ we have that

$$(i_a, i_a) \cdot (n_0(a), m_0(a)) = (i_a, i_a - n_0(a) + m_0(a)).$$

Let $W(b)$ be an open neighbourhood of b in T such that $(i_a, i_a - n_0(a) + m_0(a)) \notin W(b)$. Then the continuity of the semigroup operation in T implies that there exists an open neighbourhood $V(a)$ of a in T such that

$$V(a) \subseteq U(a) \qquad \text{and} \qquad \{(i_a, i_a)\} \cdot V(a) \subseteq W(b).$$

We fix an arbitrary element $(n,m) \in V(a) \cap \mathscr{C}_{\mathbb{Z}}$. Then we have that

$$(i_a, i_a) \cdot (n,m) = (i_a, i_a - n + m) = (i_a, i_a - n_0(a) + m_0(a)),$$

a contradiction. The obtained contradiction implies that $a \in R_{\mathscr{C}_{\mathbb{Z}}}$, and hence we have that $L_{\mathscr{C}_{\mathbb{Z}}} \subseteq R_{\mathscr{C}_{\mathbb{Z}}}$.

The proof of the inclusion $R_{\mathscr{C}_{\mathbb{Z}}} \subseteq L_{\mathscr{C}_{\mathbb{Z}}}$ is similar.

Statement (v) follows from statements $(i) - (iv)$ and Proposition 2.1(v).

(vi) Let $U(a)$ be an open neighbourhood of a in T such that conditions (a) and (b) hold, and let n_a be such integer as in condition (b). Then for all $(m_1, n_1), (m_2, n_2) \in U(a) \cap \mathscr{C}_{\mathbb{Z}}$ we have that

$$(m_1, n_1) \cdot (n_a, n_a) = (m_1 - n_1 + n_a, n_a) = (m_2 - n_2 + n_a, n_a) = (m_2, n_2) \cdot (n_a, n_a),$$

and hence the continuity of the semigroup operation in T implies that $a \in L_{\mathscr{C}_{\mathbb{Z}}}$.

(vii) Statements (i) and (iii) imply that $a \cdot (m,n) \in I$ and $(m,n) \cdot a \in I$ for all $a \in I$ and $(m,n) \in \mathscr{C}_{\mathbb{Z}}$.

Fix arbitrary elements $a, b \in I$. We consider the following two cases:

$$1)\ a \cdot b \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad 2)\ a \cdot b \in L_{\mathscr{C}_{\mathbb{Z}}}.$$

In case 1) we put $a \cdot b = (m,n) \in \mathscr{C}_{\mathbb{Z}}$. Then the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in T, respectively, such that

$$U(a) \cdot U(b) = \{(m,n)\}.$$

Since a and b are accumulation points of $\mathscr{C}_{\mathbb{Z}}$ in T, we conclude that there exist $(m_a,n_a)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$ and $(m_b,n_b)\in U(b)\cap\mathscr{C}_{\mathbb{Z}}$. Hence we have that

$$(m_a,n_a)\cdot b\in\{(m_a,n_a)\}\cdot U(b)\subseteq U(a)\cdot U(b)=\{(m,n)\}$$

and

$$a\cdot(m_b,n_b)\in U(a)\cdot\{(m_b,n_b)\}\subseteq U(a)\cdot U(b)=\{(m,n)\}$$

This implies that $a,b\in L_{\mathscr{C}_{\mathbb{Z}}}$, a contradiction.

Suppose case 2) holds and $a\cdot b=x\in L_{\mathscr{C}_{\mathbb{Z}}}$. Then by statements (i) and (iii) we have that there exist an open neighbourhood $U(x)$ of x in T and an integer n_x such that $m_1-n_1=m_2-n_2$, $m_1\leqslant n_x$ and $n_1\leqslant n_x$ for all $(m_1,n_1),(m_2,n_2)\in U(x)\cap\mathscr{C}_{\mathbb{Z}}$. Also, the continuity of the semigroup operation in T implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in T, respectively, such that

$$U(a)\cdot U(b)\subseteq U(x).$$

Since $U(a)\cap\mathscr{C}_{\mathbb{Z}}\neq\varnothing$ and $U(b)\cap\mathscr{C}_{\mathbb{Z}}\neq\varnothing$, we can find arbitrary elements $(m_a,n_a)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$ and $(m_b,n_b)\in U(b)\cap\mathscr{C}_{\mathbb{Z}}$. Then by Proposition 2.1(vi) we have that

$$x_a-y_a+m_b-n_b=m_1-n_1 \qquad\text{and}\qquad m_a-n_a+x_b-y_b=m_1-n_1$$

for all $(x_a,y_a)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$ and $(x_b,y_b)\in U(b)\cap\mathscr{C}_{\mathbb{Z}}$. This implies that there exist integers k_a and k_b such that

$$x_a-y_a=k_a \qquad\text{and}\qquad x_b-y_b=k_b$$

for all $(x_a,y_a)\in U(a)\cap\mathscr{C}_{\mathbb{Z}}$ and $(x_b,y_b)\in U(b)\cap\mathscr{C}_{\mathbb{Z}}$. Then by statement (vi) we have that $a,b\in L_{\mathscr{C}_{\mathbb{Z}}}$, a contradiction.

The obtained contradictions imply that $a\cdot b\in I$, and hence we get that the set I is an ideal of T.

$(viii)$ Proposition 2.1(vi) and assertion (vi) imply the following equalities:

$$\{x\in T\mid x\cdot(b,b)=(a,b)\}=\{x\in T\mid(a,a)\cdot x=(a,b)\}=\{x\in T\mid(a,a)\cdot x\cdot(b,b)=(a,b)\}.$$

Since by Corollary 3.2 every element (a,b) of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point in T, the continuity of the semigroup operation in T implies that ${\uparrow}(a,b)$ is a closed-and-open subset in T.

(ix) Suppose that the set ${\uparrow}(a,b)\cap L_{\mathscr{C}_{\mathbb{Z}}}$ is non-empty. Assuming that the set ${\uparrow}(a,b)\cap L_{\mathscr{C}_{\mathbb{Z}}}$ is non-singleton implies that there exist distinct $x,y\in{\uparrow}(a,b)\cap L_{\mathscr{C}_{\mathbb{Z}}}$. Then the Hausdorffness of T implies that there exist disjoint open neighbourhoods $U(x)$ and $U(y)$ of x and y in T, respectively. By the continuity of the semigroup operation in T we can find open neighbourhoods $V(1_T)$, $V(x)$ and $V(y)$ of 1_T, x and y in T, respectively, such that the following conditions hold:

$$V(x)\subseteq U(x),\quad V(y)\subseteq U(y),\quad V(1_T)\cdot V(x)\subseteq U(x)\quad\text{and}\quad V(1_T)\cdot V(y)\subseteq U(y).$$

By assertions $(i)-(iii)$ we can find the integers n,n_1,n_2,m_1 and m_2 such that

$$(n,n)\in V(1_T),\quad (n_1,n_2)\in V(x),\quad (m_1,m_2)\in V(y),\quad n_1-n_2=m_1-m_2,$$
$$n\geqslant n_1\quad\text{and}\quad n\geqslant m_1.$$

Then we have that

$$(n,n)\cdot(n_1,n_2)=(n,n-n_1+n_2)=(n,n-m_1+m_2)=(n,n)\cdot(m_1,m_2),$$

and hence $(V(1_T)\cdot V(x))\cdot(V(1_T)\cdot V(y))\neq\varnothing$, a contradiction. The obtained contradiction implies that $x=y$.

(x) Statement (vii) implies that $T\setminus(I\cup\mathscr{C}_{\mathbb{Z}})=L_{\mathscr{C}_{\mathbb{Z}}}$. Let $\mathbb{Z}$ be the additive group of integers. We define a map $\mathfrak{h}\colon L_{\mathscr{C}_{\mathbb{Z}}}\to\mathbb{Z}$ as follows:

$$(x)\mathfrak{h}=n\qquad\text{if and only if there exists a neighbourhood } U(x) \text{ of } x \text{ in } T \text{ such that}$$
$$a-b=n,\ \text{for all } (a,b)\in U(x)\cap\mathscr{C}_{\mathbb{Z}},$$

where $x\in L_{\mathscr{C}_{\mathbb{Z}}}$. We observe that assertions $(i)-(v)$ imply that the map $\mathfrak{h}$ is well defined. Also, Proposition 2.1 implies that $\mathfrak{h}\colon L_{\mathscr{C}_{\mathbb{Z}}}\to\mathbb{Z}$ is a monomorphism, and hence $L_{\mathscr{C}_{\mathbb{Z}}}$ is a submonoid of $\mathbb{Z}$.

In the case when a maximal subgroup of $L_{\mathscr{C}_{\mathbb{Z}}}$ is non-singleton Proposition 3.7 implies that $(L_{\mathscr{C}_{\mathbb{Z}}})\mathfrak{h}$ is a cyclic subgroup of $\mathbb{Z}$. This completes the proof of our assertion.

(xi) Assertion (v) implies that

$$\uparrow\mathscr{C}_{\mathbb{Z}} = \{x \in T \mid \text{there exists } y \in \mathscr{C}_{\mathbb{Z}} \text{ such that } x \cdot y \in \mathscr{C}_{\mathbb{Z}}\} = \bigcup_{(a,b)\in\mathscr{C}_{\mathbb{Z}}} \uparrow(a,b).$$

Then assertion $(viii)$ implies that $\uparrow\mathscr{C}_{\mathbb{Z}}$ is an open subset in T and hence by assertion (vii) we get that the ideal I is a closed subset of T.

Assertion (xii) follows from (x). □

4. On a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup

For every non-negative integer k by $k\mathbb{Z}$ we denote a subgroup of the additive group of integers $\mathbb{Z}$ which is generated by an element $k \in \mathbb{Z}$. We observe if $k = 0$ then the group $k\mathbb{Z}$ is trivial. Also, we denote $G_0 = \mathbb{Z}$ and $G_1(k) = k\mathbb{Z}$ for a positive integer k.

The following five examples illustrate distinct structures of a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup.

Example 4.1. Let be $S_1 = G_1(0) \sqcup \mathscr{C}_{\mathbb{Z}}$. Then $G_1(0)$ is a trivial group and we put $\{e_1\} = G_1(0)$. We extend the semigroup operation from $\mathscr{C}_{\mathbb{Z}}$ onto S_1 as follows:

$$e_1 \cdot (a,b) = (a,b) \cdot e_1 = (a,b) \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad e_1 \cdot e_1 = e_1,$$

i.e., S_1 is the semigroup $\mathscr{C}_{\mathbb{Z}}$ with the adjoined unit e_1. We fix an arbitrary decreasing sequence $\{m_i\}_{i\in\mathbb{N}}$ of negative integers and for every positive integer n we put

$$U_n(e_1) = \{e_1\} \cup \{(m_i, m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant n\}.$$

Then we determine a topology τ_1 on S_1 as follows:

1) all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_1, τ_1); and
2) the family $\mathscr{B}_1(e_1) = \{U_n(e_1) \mid n \in \mathbb{N}\}$ is a base of the topology τ_1 at the point $e_1 \in G_1(0) \subseteq S_1$.

Then for every positive integer n we have that

$$U_n(e_1) \cdot U_n(e_1) = U_n(e_1) \qquad \text{and} \qquad (U_n(e_1))^{-1} = U_n(e_1).$$

Let (m,n) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$. We fix a positive integer $i_{(m,n)}$ such that $m_{i_{(m,n)}} \leqslant m$ and $m_{i_{(m,n)}} \leqslant n$. Then we have that

$$U_{i_{(m,n)}}(e_1) \cdot \{(m,n)\} = \{(m,n)\} \qquad \text{and} \qquad \{(m,n)\} \cdot U_{i_{(m,n)}}(e_1) = \{(m,n)\}.$$

Hence we get that (S_1, τ_1) is a topological inverse semigroup. Obviously, (S_1, τ_1) is a Hausdorff locally compact space.

Example 4.2. Let k and n be any positive integers such that $n \in \{1, \ldots, k\}$ is a divisor of k and we put $k = n \cdot s$, where s is some positive integer. We put $S_2 = G_1(k) \sqcup \mathscr{C}_{\mathbb{Z}}$. Later an element of the group $G_1(k) = k\mathbb{Z}$ will be denote by ki, where $i \in \mathbb{Z}$. We extend the semigroup operation from $\mathscr{C}_{\mathbb{Z}}$ onto S_2 by the following way:

$$ki \cdot (a,b) = (-ki + a, b) \in \mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad (a,b) \cdot ki = (a, b + ki) \in \mathscr{C}_{\mathbb{Z}},$$

for arbitrary $(a,b) \in \mathscr{C}_{\mathbb{Z}}$ and $ki \in G_1(k)$. To see that the extended binary operation is associative we need only check six possibilities, the other being evident.

Then for arbitrary $ki_1, ki_2 \in G_1(k)$ and $(a,b), (c,d) \in \mathscr{C}_{\mathbb{Z}}$ we have that:

1) $(ki_1 \cdot ki_2) \cdot (a,b) = (ki_1 + ki_2)(a,b) = (-ki_1 - ki_2 + a, b) = ki_1 \cdot (-ki_2 + a, b) = ki_1 \cdot (ki_2 \cdot (a,b))$;
2) $(a,b) \cdot (ki_1 \cdot ki_2) = (a,b) \cdot (ki_1 + ki_2) = (a, b + ki_1 + ki_2) = (a, b + ki_1) \cdot ki_2 = ((a,b) \cdot ki_1) \cdot ki_2$;
3) $(ki_1 \cdot (a,b)) \cdot ki_2 = (-ki_1 + a, b) \cdot ki_2 = (-ki_1 + a, b + ki_2) = ki_1 \cdot (a, b + ki_2) = ki_1 \cdot ((a,b) \cdot ki_2)$;

4) $(ki_1\cdot(a,b))\cdot(c,d)=(-ki_1+a,b)\cdot(c,d)=\begin{cases}(-ki_1+a-b+c,d), & \text{if } b\leqslant c;\\ (-ki_1+a,b-c+d), & \text{if } b\geqslant c\end{cases}$

$=\begin{cases}ki_1\cdot(a-b+c,d), & \text{if } b\leqslant c;\\ ki_1\cdot(a,b-c+d), & \text{if } b\geqslant c\end{cases}=ki_1\cdot((a,b)\cdot(c,d));$

5) $((a,b)\cdot(c,d))\cdot ki_1=\begin{cases}(a-b+c,d)\cdot ki_1, & \text{if } b\leqslant c;\\ (a,b-c+d)\cdot ki_1, & \text{if } b\geqslant c\end{cases}$

$=\begin{cases}(a-b+c,d+ki_1), & \text{if } b\leqslant c;\\ (a,b-c+d+ki_1), & \text{if } b\geqslant c\end{cases}=(a,b)\cdot(c,d+ki_1)=(a,b)\cdot((c,d)\cdot ki_1);$

6) $((a,b)\cdot ki_1)\cdot(c,d)=(a,b+ki_1)\cdot(c,d)=\begin{cases}(a-b-ki_1+c,d), & \text{if } b+ki_1\leqslant c;\\ (a,b+ik_1-c+d), & \text{if } b+ki_1\geqslant c\end{cases}$

$=\begin{cases}(a-b-ki_1+c,d), & \text{if } b\leqslant -ki_1+c;\\ (a,b+ki_1-c+d), & \text{if } b\geqslant -ki_1+c\end{cases}=(a,b)\cdot(-ki_1+c,d)=(a,b)\cdot(ki_1\cdot(c,d)).$

Also simple verifications show that S_2 is an inverse semigroup.

Let ki be an arbitrary element of the group $G_1(k)$. For every positive integer j we denote

$$U_j^n(ki)=\{ki\}\cup\{(-nq,-nq+ki)\mid q\geqslant j, q\in\mathbb{N}\}.$$

We determine a topology τ_2 on S_2 as follows:

1) all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_2,τ_2); and
2) the family $\mathscr{B}_2(ki)=\left\{U_j^n(ki)\mid j\in\mathbb{N}\right\}$ is a base of the topology τ_2 at the point $ki\in G_1(k)\subseteq S_2$.

Then for every positive integer j we have that

$$U_j^n(ki_1)\cdot U_{j-i_1s}^n(ki_2)\subseteq U_j^n(ki_1+ki_2) \qquad \text{and} \qquad \left(U_j^n(ki_1)\right)^{-1}=U_j^n(-ki_1),$$

for $ki_1,ki_2\in G_1(k)$.

Let (a,b) be an arbitrary element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ and $ki\in G_1(k)$. Then we have that

$$U_j^n(ki)\cdot\{(a,b)\}=\{(a-ki,b)\} \qquad \text{and} \qquad \{(a,b)\}\cdot U_j^n(ki)=\{(a,b+ki)\},$$

for every positive integer j such that $nj\geqslant\max\{-b;ki-a\}$.

Therefore (S_2,τ_2) is a topological inverse semigroup, and moreover the topological space (S_2,τ_2) is Hausdorff and locally compact.

Example 4.3. We put $S_3=\mathscr{C}_{\mathbb{Z}}\sqcup G_0$ and extend the semigroup operation from the semigroup $\mathscr{C}_{\mathbb{Z}}$ onto S_3 by the following way:

$$(a,b)\cdot n=n\cdot(a,b)=n+b-a\in G_0,$$

for all $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ and $n\in G_0$. To see that the extended binary operation is associative we need only check two possibilities, the other being evident.

Then for arbitrary $m,n\in G_0$ and $(a,b),(c,d)\in\mathscr{C}_{\mathbb{Z}}$ we have that:

1) $(n\cdot(a,b))\cdot(c,d)=(n+b-a)\cdot(c,d)=n+b-a+d-c=\begin{cases}n\cdot(a-b+c,d), & \text{if } b\leqslant c;\\ n\cdot(a,b-c+d), & \text{if } b\geqslant c\end{cases}$
$=n\cdot((a,b)\cdot(c,d));$
2) $(m\cdot n)\cdot(a,b)=m+n+b-a=m\cdot(n+b-a)=m\cdot(n\cdot(a,b)).$

This completes the proof of the associativity of such defined binary operation on S_3. Also, we observe that S_3 with such defined semigroup operation is an inverse semigroup.

For every positive integer n and every element $k\in G_0$ we put:

$$U_n(k)=\begin{cases}\{k\}\cup\{(a,a+k)\mid a=n,n+1,n+2,\ldots\}, & \text{if } k\geqslant 0;\\ \{k\}\cup\{(a-k,a)\mid a=n,n+1,n+2,\ldots\}, & \text{if } k\leqslant 0.\end{cases}$$

We determine a topology τ_3 on S_3 as follows:

1) all elements of the semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated points in (S_3,τ_3); and
2) the family $\mathscr{B}_3(k)=\{U_n(k)\mid n\in\mathbb{N}\}$ is a base of the topology τ_3 at the point $k\in G_0\subseteq S_3$.

Then for all $k_1,k_2\in G_0$ we have that

$$U_{2n}(k_1)\cdot U_{2n}(k_2)\subseteq U_n(k_1+k_2),$$

for every positive integer $n\geqslant\max\{|k_1|,|k_2|\}$, and

$$(U_i(k_1))^{-1}=U_i(-k_1),$$

for every positive integer i. Also, for arbitrary $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ and $k\in G_0$ we have that

$$(a,b)\cdot U_{2n}(k)\subseteq U_n(k+b-a) \qquad \text{and} \qquad U_{2n}(k)\cdot(a,b)\subseteq U_n(k+b-a),$$

for every positive integer $n\geqslant\max\{|a|,|b|,|k|\}$.

This completes the proof that (S_3,τ_3) is a topological inverse semigroup. Obviously, (S_3,τ_3) is a Hausdorff locally compact space.

Example 4.4. Let be $S_4=G_1(0)\sqcup S_3$, where the group $G_1(0)$ and the semigroup S_3 are defined in Example 4.1 and Example 4.3, respectively. We extend the semigroup operation from S_3 onto S_4 as follows:

$$e_1\cdot x=x\cdot e_1=x\in\mathscr{C}_{\mathbb{Z}} \qquad \text{and} \qquad e_1\cdot e_1=e_1,$$

i.e., S_4 is the semigroup S_3 with the adjoined unit e_1.

Let τ_4 be a topology on S_4 which is generated by the family $\tau_1\cup\tau_3$ (see Examples 4.1 and 4.3). Then for every element $k_0\in G_0$ and every positive integers n_1 and n_0 we have that the following inclusions hold:

$$U_{n_1}(e_1)\cdot U_{n_0}(k_0)\subseteq U_{n_0}(k_0) \qquad \text{and} \qquad U_{n_0}(k_0)\cdot U_{n_1}(e_1)\subseteq U_{n_0}(k_0),$$

where $U_{n_1}(e_1)\in\mathscr{B}_1(e_1)$ and $U_{n_0}(k_0)\in\mathscr{B}_3(k_0)$ (see Examples 4.1 and 4.3). These inclusions and Examples 4.1 and 4.3 imply that (S_4,τ_4) is a Hausdorff topological inverse semigroup. Obviously, (S_4,τ_4) is a locally compact space.

Example 4.5. Let k and n be such positive integers as in Example 4.2. We put $S_5=G_1(k)\sqcup\mathscr{C}_{\mathbb{Z}}\sqcup G_0$ and extend semigroup operation from S_2 and S_3 onto S_5 as follows. Later we denote elements of groups $G_1(K)$ and G_0 by $(ki)^1$ and $(n)^0$, respectively. We put

$$(ki)^1\cdot(n)^0=(n)^0\cdot(ki)^1=(ki+n)^0\in G_0,$$

for all $(ki)^1\in G_1(k)$ and $(n)^0\in G_0$. To see that the extended binary operation is associative we need only check twelve possibilities, the other either are evident or are proved in Examples 4.2 and 4.3.

Then for arbitrary $(ki_1)^1,(ki_2)^1\in G_1(k)$, $(n_1)^0,(n_2)^0\in G_0$ and $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ we have that:

1) $((n_1)^0\cdot(n_2)^0)\cdot(ki_1)^1=(n_1+n_2)^0\cdot(ki_1)^1=(n_1+n_2+ki_1)^0=(n_1)^0\cdot(n_2+ki_1)^0$ $=(n_1)^0\cdot((n_2)^0\cdot(ki_1)^1)$;
2) $((n_1)^0\cdot(ki_1)^1)\cdot(n_2)^0=(n_1+ki_1)^0\cdot(n_2)^0=(n_1+ki_1+n_2)^0=(n_1)^0\cdot(ki_1+n_2)^0$ $=(n_1)^0\cdot((ki_1)^1\cdot(n_2)^0)$;
3) $((n_1)^0\cdot(ki_1)^1)\cdot(ki_2)^1=(n_1+ki_1)^0\cdot(ki_2)^1=(n_1+ki_1+ki_2)^0=(n_1)^0\cdot(ki_1+ki_2)^1$ $=(n_1)^0\cdot((ki_1)^1\cdot(ki_2)^1)$;
4) $((n_1)^0\cdot(ki_1)^1)\cdot(a,b)=(n_1+ki_1)^0\cdot(a,b)=(n_1+ki_1+b-a)^0=(n_1)^0\cdot(-ki_1+a,b)$ $=(n_1)^0\cdot((ki_1)^1\cdot(a,b))$;
5) $((n_1)^0\cdot(a,b))\cdot(ki_1)^1=(n_1+b-a)^0\cdot(ki_1)^1=(n_1+b-a+ki_1)^0=(n_1)^0\cdot(a,b+ki_1)$ $=(n_1)^0\cdot((a,b)\cdot(ki_1)^1)$;
6) $((ki_1)^1\cdot(n_1)^0)\cdot(n_2)^0=(ki_1+n_1)^0\cdot(n_2)^0=(ki_1+n_1+n_2)^0=(ki_1)^1\cdot(n_1+n_2)^0$ $=(ki_1)^1\cdot((n_1)^0\cdot(n_2)^0)$;
7) $((ki_1)^1\cdot(n_1)^0)\cdot(ki_2)^1=(ki_1+n_1)^0\cdot(ki_2)^1=(ki_1+n_1+ki_2)^0=(ki_1)^1\cdot(n_1+ki_2)^0$ $=(ki_1)^1\cdot((n_1)^0\cdot(ki_2)^1)$;
8) $((ki_1)^1\cdot(n_1)^0)\cdot(a,b)=(ki_1+n_1)^0\cdot(a,b)=(ki_1+n_1+b-a)^0=(ki_1)^1\cdot(n_1+b-a)^0$ $=(ki_1)^1\cdot((n_1)^0\cdot(a,b))$;

9) $((ki_1)^1 \cdot (ki_2)^1) \cdot (n_1)^0 = (ki_1 + ki_2)^1 \cdot (n_1)^0 = (ki_1 + ki_2 + n_1)^0 = (ki_1)^1 \cdot (ki_2 + n_1)^0$
$= (ki_1)^1 \cdot ((ki_2)^1 \cdot (n_1)^0)$;
10) $((ki_1)^1 \cdot (a,b)) \cdot (n_1)^0 = (-ki_1 + a, b) \cdot (n_1)^0 = (ki_1 + b - a + n_1)^0$
$= (ki_1)^1 \cdot (b - a + n_1)^0 = (ki_1)^1 \cdot ((a,b) \cdot (n_1)^0)$;
11) $((a,b) \cdot (n_1)^0) \cdot (ki_1)^1 = (b - a + n_1)^0 \cdot (ki_1)^1 = (b - a + n_1 + ki_1)^0 = (a,b) \cdot (n_1 + ki_1)^0$
$= (a,b) \cdot ((n_1)^0 \cdot (ki_1)^1)$;
12) $((a,b) \cdot (ki_1)^1) \cdot (n_1)^0 = (a, b + ki_1)^0 \cdot (n_1)^0 = (b + ki_1 - a + n_1)^0 = (a,b) \cdot (ki_1 + n_1)^0$
$= (a,b) \cdot ((ki_1)^1 \cdot (n_1)^0)$.

This completes the proof of the associativity of such defined binary operation on S_5. Also, we observe that S_5 with such defined semigroup operation is an inverse semigroup.

Let τ_5 be a topology on S_5 which is generated by the family $\tau_2 \cup \tau_3$ (see Examples 4.2 and 4.3). Also Examples 4.2 and 4.3 imply that it is sufficient to show that the semigroup operation in S_5 is continuous in cases $(ki)^1 \cdot (n)^0$ and $(n)^0 \cdot (ki)^1$, where $(n)^0 \in G_0$ and $(ki)^1 \in G_1(k)$. Then for every positive integer $p \geqslant \max\{|ki|, |n|\}$ we have that

$$U_{2p}\big((ki)^1\big) \cdot U_{2p}\big((n)^0\big) \subseteq U_p\big((ki+n)^0\big) \qquad \text{and} \qquad U_{2p}\big((n)^0\big) \cdot U_{2p}\big((ki)^1\big) \subseteq U_p\big((ki+n)^0\big).$$

This completes the proof that (S_5, τ_5) is a topological inverse semigroup. Obviously, (S_5, τ_5) is a locally compact space.

Theorem 4.6. *Let T be a Hausdorff topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup and $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$, then the following assertions hold:*

(i) $E(T)$ is a countable linearly ordered semilattice;
(ii) $E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$ is a singleton set;
(iii) $T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is a subgroup in T.

Proof. (*i*) By Proposition II.3 from [8] we have that $\operatorname{cl}_T(E(\mathscr{C}_{\mathbb{Z}})) = E(T)$ and since the closure of a linearly ordered subsemilattice in a topological semilattice is a linearly ordered subsemilattice too (see [12, Lemma 1]) we get that $E(T)$ is a linearly ordered semilattice. Then the semilattice operation in $E(T)$ implies that the sets $E(T) \setminus \bigcup\limits_{e \in E(\mathscr{C}_{\mathbb{Z}})} {\downarrow} e$ and $E(T) \setminus \bigcup\limits_{e \in E(\mathscr{C}_{\mathbb{Z}})} {\uparrow} e$ are either singleton or empty. This completes the proof of our assertion.

Assertion (*ii*) follows from assertion (*i*).

(*iii*) Since T is an inverse semigroup and $\overline{e}$ is a minimal idempotent in $E(T)$ we conclude that the $\mathscr{H}$-class $H_{\overline{e}}$ which contains $\overline{e}$ coincides with the ideal $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$. Indeed, if there exist $x \in I$ and an $\mathscr{H}$-class $H_x \subseteq I$ in T such that $x \in H_x \neq H_{\overline{e}}$, then since T is an inverse semigroup we have that there exists an idempotent $e \in T$ such that either $xx^{-1} = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$ or $x^{-1}x = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$. If $xx^{-1} = e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$, then we have that $x = xx^{-1}x = ex \in eT$, and since T is an inverse semigroup Theorem 1.17 from [7] implies $e \in xT$, a contradiction. Similar arguments show that $x^{-1}x \neq e \in {\uparrow}\mathscr{C}_{\mathbb{Z}}$. Hence assertion (*ii*) implies that $xx^{-1} = x^{-1}x = \overline{e}$ and hence $x \in H_x = H_{\overline{e}}$. $\square$

The following theorem describes the structure of a closure of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a locally compact topological inverse semigroup T, i.e., it gives the description of the non-empty ideal $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ in the remainder of $\mathscr{C}_{\mathbb{Z}}$ in T.

Theorem 4.7. *Let T be a Hausdorff locally compact topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup and $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}} \neq \varnothing$, then the following assertions hold:*

(i) ${\downarrow} e_n$ is a compact subsemilattice in $E(T)$ for every idempotent $e_n = (n,n) \in \mathscr{C}_{\mathbb{Z}}$, $n \in \mathbb{Z}$;
(ii) $T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is isomorphic to the discrete additive group of integers;
(iii) if $\overline{e}$ is a unit of $T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$, then the map $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ which is defined by the formula $((a,b))\mathfrak{h} = (a,b) \cdot \overline{e}$ is the natural homomorphism generated by the minimal group congruence $\mathfrak{C}_{mg}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$;

(*iv*) *the subsemigroup* $S = \mathscr{C}_{\mathbb{Z}} \cup I$ *is topologically isomorphic to the topological inverse semigroup* (S_3, τ_3) *from Example 4.3.*

Proof. (*i*) We show that ${\downarrow}e_0$ is a compact subset in $E(T)$ for $e_0 = (0,0)$. By assertion (*ii*) of Theorem 4.6 we get that the set $E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$ is singleton and we put $\{\overline{e}\} = E(T) \cap (T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}})$. Then $\overline{e}$ is a smallest idempotent in $E(T)$. By Theorem 1.5 from [6, Vol. 1] we have that $E(T)$ is a closed subset in T, and hence by Theorem 3.3.9 from [9] we get that $E(T)$ is a locally compact space. Suppose the contrary: ${\downarrow}e_0$ is not a compact subset in $E(T)$. Since Corollary 3.2 implies that every element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point in T and hence so it is in $E(T)$, we get that there exists an open neighbourhood $U(\overline{e})$ of $\overline{e}$ in $E(T)$ such that the set ${\downarrow}e_0 \setminus U(\overline{e})$ is an infinite discrete subspace of $E(T)$, $U(\overline{e}) \subseteq E(T) \setminus {\uparrow}e_0$ and $\operatorname{cl}_{E(T)}(U(\overline{e})) = U(\overline{e})$ is a compact subset of $E(T)$. Then for every positive integer i there exists an integer $j \geqslant i$ such that $(j,j) \notin U(\overline{e})$ and $(j+1, j+1) \in U(\overline{e})$. Then the semigroup operation in $\mathscr{C}_{\mathbb{Z}}$ implies that by induction we can construct an infinite subset $M \subseteq {\downarrow}e_0 \setminus \{\overline{e}\}$ of $E(T)$ such that $M \subseteq U(\overline{e}) \setminus \{\overline{e}\}$ and $\{(0,1)\} \cdot M \cdot \{(1,0)\} \subseteq {\downarrow}e_0 \setminus U(\overline{e})$. Since the set $U(\overline{e})$ is compact and the set $M \subseteq U(\overline{e}) \setminus \{\overline{e}\}$ contains only isolated points from $E(\mathscr{C}_{\mathbb{Z}})$, we conclude that $\overline{e} \in \operatorname{cl}_T(M)$. Since ${\downarrow}e_0 \setminus U(\overline{e})$ is a closed subset of $E(T)$ we have that the continuity of the semigroup operation in T and Proposition 1.4.1 from [9] imply that

$$\overline{e} \in \{(0,1)\} \cdot \operatorname{cl}_T(M) \cdot \{(1,0)\} \subseteq \operatorname{cl}_T\left(\{(0,1)\} \cdot M \cdot \{(1,0)\}\right) \subseteq {\downarrow}e_0 \setminus U(\overline{e}),$$

which contradicts $\overline{e} \in U(\overline{e})$. The obtained contradiction implies that the set ${\downarrow}e_0 \setminus U(\overline{e})$ is finite, and hence the set ${\downarrow}e_0$ is compact. Since for every integer n the set ${\downarrow}e_n \setminus {\downarrow}e_0$ is either finite or empty and e_n is an isolated point in $E(T)$ we conclude that ${\downarrow}e_n$ is a compact subsemilattice of $E(T)$.

(*ii*) By assertion (*i*) we have that $\overline{e}$ is an accumulation point of the subsemigroup $\mathscr{C}_{\mathbb{N}}[0]$ in T. Since by Theorem 3.3.9 from [9] a closed subset of a locally compact space is a locally compact subspace too, and by Proposition 2.1(*viii*) the semigroup $\mathscr{C}_{\mathbb{N}}[0]$ is isomorphic to the bicyclic semigroup, Proposition V.3 from [8] implies that the subset $\operatorname{cl}_T(\mathscr{C}_{\mathbb{N}}[0]) \setminus \mathscr{C}_{\mathbb{N}}[0]$ is a non-singleton subgroup of T. By Corollary 3.2 we get that $\mathscr{C}_{\mathbb{Z}}$ is an open discrete subsemigroup of T and hence we get that $\operatorname{cl}_T(\mathscr{C}_{\mathbb{N}}[0]) \setminus \mathscr{C}_{\mathbb{N}}[0] \subseteq \operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}}) \setminus \mathscr{C}_{\mathbb{Z}}$.

By assertion (*iii*) of Theorem 4.6 we have that $I = T \setminus {\uparrow}\mathscr{C}_{\mathbb{Z}}$ is a non-singleton subgroup in T. Since T is a topological inverse semigroup we get that I is a topological group. Then by Proposition 3.8(*xi*) we have that I is a closed subset of T and hence by Theorem 3.3.9 from [9] we get that I is a locally compact topological group.

Later we show that $(a,b) \cdot \overline{e} = \overline{e} \cdot (a,b)$ for every $(a,b) \in \mathscr{C}_{\mathbb{Z}}$. Suppose the contrary: there exists $(a,b) \in \mathscr{C}_{\mathbb{Z}}$ such that $(a,b) \cdot \overline{e} \neq \overline{e} \cdot (a,b)$. Without loss of generality we can assume that $a \leqslant b$ in $\mathbb{Z}$. Then the Hausdorffness of the space T implies that there exist open neighbourhoods $U((a,b)\cdot\overline{e})$ and $U(\overline{e}\cdot(a,b))$ of the points $(a,b)\cdot\overline{e}$ and $\overline{e}\cdot(a,b)$ in T such that $U((a,b)\cdot\overline{e}) \cap U(\overline{e}\cdot(a,b)) = \varnothing$. Then the continuity of the semigroup operation of T implies that there exists an open neighbourhood $V(\overline{e})$ of $\overline{e}$ in T such that the following conditions hold:

$$\{(a,b)\} \cdot V(\overline{e}) \subseteq U((a,b)\cdot\overline{e}) \qquad \text{and} \qquad V(\overline{e}) \cdot \{(a,b)\} \subseteq U(\overline{e}\cdot(a,b)).$$

By assertion (*i*) we get that without loss of generality we can assume that $V(\overline{e}) \cap E(T)$ is a compact subset in T and there exists a positive integer $n_0 \geqslant \max\{a,b\}$ such that $(n,n) \in V(\overline{e}) \cap E(T)$ for all integers $n \geqslant n_0$. Then for $n = 2n_0 - a$ and $k = 2n_0 - b$ we get that $(n,n),(k,k) \in V(\overline{e}) \cap E(T)$. But we have

$$(a,b)\cdot(n,n) = (a,b)\cdot(2n_0 - a, 2n_0 - a) = (2n_0 - a - b + a, 2n_0 - a) = (2n_0 - b, 2n_0 - a)$$

and

$$(k,k)\cdot(a,b) = (2n_0 - b, 2n_0 - b)\cdot(a,b) = (2n_0 - b, 2n_0 - b - a + b) = (2n_0 - b, 2n_0 - a),$$

which contradicts $U((a,b)\cdot\overline{e}) \cap U(\overline{e}\cdot(a,b)) = \varnothing$. The obtained contradiction implies that $(a,b)\cdot\overline{e} = \overline{e}\cdot(a,b)$ for every $(a,b) \in \mathscr{C}_{\mathbb{Z}}$.

Next we show that $x\cdot\overline{e}=\overline{e}\cdot x$ for every $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$. Suppose contrary: there exists $x\in T\setminus\mathscr{C}_{\mathbb{Z}}$ such that $x\cdot\overline{e}\neq\overline{e}\cdot x$. Then the Hausdorffness of the space T implies that there exist open neighbourhoods $U(x\cdot\overline{e})$ and $U(\overline{e}\cdot x)$ of the points $x\cdot\overline{e}$ and $\overline{e}\cdot x$ in T such that $U(x\cdot\overline{e})\cap U(\overline{e}\cdot x)=\varnothing$. The continuity of the semigroup operation of T implies that there exists an open neighbourhood $V(x)$ of x in T such that the following conditions hold:

$$V(x)\cdot\{\overline{e}\}\subseteq U(x\cdot\overline{e}) \qquad \text{and} \qquad \{\overline{e}\}\cdot V(x)\subseteq U(\overline{e}\cdot x).$$

Since $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T we conclude that there exists $(a,b)\in\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)\in V(x)$. Then we get that $(a,b)\cdot\overline{e}=\overline{e}\cdot(a,b)$, which contradicts $U(x\cdot\overline{e})\cap U(\overline{e}\cdot x)=\varnothing$. The obtained contradiction implies that $x\cdot\overline{e}=\overline{e}\cdot x$ for every $x\in T$.

We define a map $\mathfrak{h}\colon T\to I$ by the formula $(x)\mathfrak{h}=x\cdot\overline{e}$. Since $x\cdot\overline{e}=\overline{e}\cdot x$ for every $x\in T$ we get that $\mathfrak{h}$ is a homomorphism. Since $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T, Proposition 2.2 and assertion (iii) of Theorem 4.6 imply that the topological group I contains a dense cyclic subgroup. Since I is a locally compact topological group, Pontryagin-Weil Theorem (see [15, p. 71, Theorem 19]) implies that either I is compact or I is discrete. If I is compact, then by Proposition 3.8$(viii)$ we get that

$$S=T\setminus\bigcup_{(a,b)\notin\mathscr{C}_{\mathbb{N}}[0]}{\uparrow}(a,b)$$

is a closed subset in T. Then by Theorem 3.3.9 from [9] S is a locally compact space. Obviously, $S=\mathscr{C}_{\mathbb{N}}[0]\cup I$. Since I is a locally compact ideal in T, Proposition 2.1$(viii)$ and Proposition II.4 from [8] imply that the Rees quotient semigroup S/I with the quotient topology is locally compact topological inverse semigroup which is isomorphic to the bicyclic semigroup with an adjoined zero. This contradicts Proposition V.3 from [8]. The obtained contradiction implies that the group I is discrete and hence I is a discrete additive group of integers.

(iii) Let $(a,b),(c,d)\in\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)\mathfrak{C}_{mg}(c,d)$. Then there exists an idempotent $(n,n)\in\mathscr{C}_{\mathbb{Z}}$ such that $(a,b)\cdot(n,n)=(c,d)\cdot(n,n)$. Since $(i,i)\cdot\overline{e}=\overline{e}$ for every idempotent $(i,i)\in\mathscr{C}_{\mathbb{Z}}$ we get that $((a,b))\mathfrak{h}=((c,d))\mathfrak{h}$.

Let $(a,b),(c,d)\in\mathscr{C}_{\mathbb{Z}}$ such that $((a,b))\mathfrak{h}=((c,d))\mathfrak{h}$. Suppose the contrary: $(a,b)\cdot(n,n)\neq(c,d)\cdot(n,n)$ for any idempotent $(n,n)\in\mathscr{C}_{\mathbb{Z}}$. If $(a,b)\cdot(n_1,n_1)=(c,d)\cdot(n_2,n_2)$ for some idempotents $(n_1,n_1),(n_2,n_2)\in\mathscr{C}_{\mathbb{Z}}$, then we have that

$$\begin{aligned}
(a,b)\cdot(n_1,n_1)\cdot(n_2,n_2)&=(a,b)\cdot(n_1,n_1)\cdot(n_1,n_1)\cdot(n_2,n_2)\\
&=(c,d)\cdot(n_2,n_2)\cdot(n_1,n_1)\cdot(n_2,n_2)\\
&=(c,d)\cdot(n_1,n_1)\cdot(n_2,n_2).
\end{aligned}$$

Therefore we get that $(a,b)\cdot(n_1,n_1)\neq(c,d)\cdot(n_2,n_2)$ for all idempotents $(n_1,n_1),(n_2,n_2)\in\mathscr{C}_{\mathbb{Z}}$. Then Proposition 2.1$(vi)$ implies that $b-a\neq d-c$, and hence by the proof of Proposition 2.2 we get that the congruence on the semigroup $\mathscr{C}_{\mathbb{Z}}$ which is generated by the homomorphism $\mathfrak{h}$ distincts from the minimal group congruence $\mathfrak{C}_{mg}$ on $\mathscr{C}_{\mathbb{Z}}$. Then the ideal I is not isomorphic to the additive group of integers $\mathbb{Z}$ and hence by Proposition 2.2 we have that the ideal I contains a finite cyclic group. This contradicts assertion (ii). The obtained contradiction implies our assertion.

(iv) Assertions (ii) and (iii) imply that the subsemigroup $S=\mathscr{C}_{\mathbb{Z}}\cup I$ of T is algebraically isomorphic to the inverse semigroup S_3 from Example 4.3. We identify the group I with G_0 and put $\overline{e}=0\in G_0$.

By τ we denote the topology of the topological inverse semigroup T. Since G_0 is a discrete subgroup of T, assertion (i) implies that there exists a compact open neighbourhood $U(0)$ of 0 in T with the following property:

> $U(0)\subseteq E(T)$ and there is a positive integer n_0 such that $n_0=\max\{(n,n)\in E(\mathscr{C}_{\mathbb{Z}})\mid (n,n)\in U(0)\}$ and $(i,i)\in U(0)$ for all integers $i\geqslant n_0$.

Hence, we get that $\mathscr{B}_3(0)=\{U_n(0)\mid n\in\mathbb{N}\}$ is a base of the topology of the space T at the point $0\in G_0\subseteq T$, where $U_n(0)=\{0\}\cup\{(n+i,n+i)\mid i\in\mathbb{N}\}$.

We fix an arbitrary element $k \in G_0$. Without loss of generality we can assume that $k \geqslant 0$. Then $k^{-1} = -k \in \mathbb{Z} = G_0$. Since G_0 is a discrete subgroup of T, the continuity of the homomorphism $\mathfrak{h}\colon T \to G_0\colon x \mapsto x \cdot \overline{e} = x \cdot 0$ implies that $(k)\mathfrak{h}^{-1}$ is an open subset in T. We observe that, since the homomorphism $\mathfrak{h}$ generates the minimal group congruence on $\mathscr{C}_{\mathbb{Z}}$ (see assertion (iii)) we get that $(k)\mathfrak{h}^{-1} \cap \mathscr{C}_{\mathbb{Z}} = \{(a,b) \in \mathscr{C}_{\mathbb{Z}} \mid b-a=k\}$. Also, since

$$\uparrow(a,b) = \{(x,y) \in \mathscr{C}_{\mathbb{Z}} \mid (x,y)\cdot(b,b) = (a,b)\},$$

for every $(a,b) \in \mathscr{C}_{\mathbb{Z}}$, Proposition 3.8$(viii)$ implies that $\uparrow(a,b)$ is a closed-and-open subset in T for every $(a,b) \in \mathscr{C}_{\mathbb{Z}}$. Hence we get that $\{k\} \cup \{(i,i+k) \in \mathscr{C}_{\mathbb{Z}} \mid i = 1,2,3,\ldots\}$ is an open subset in T.

We fix an arbitrary positive integer i. Since $(i+k,i)\cdot k = 0 \in G_0$, the continuity of the semigroup operation in T implies that for every $U_i(0) \in \mathscr{B}_3(0)$ there exists an open neighbourhood

$$V(k) \subseteq \{k\} \cup \{(i,i+k) \in \mathscr{C}_{\mathbb{Z}} \mid i = 1,2,3,\ldots\}$$

of k in T such that $(i+k,i)\cdot V(k) \subseteq U_i(0)$. Then the semigroup operation of $\mathscr{C}_{\mathbb{Z}}$ implies that $V(k) \subseteq U_i(k)$ for $U_i(k) \in \mathscr{B}_3(k)$.

We observe that for every $k \in G_0$ and for every positive integer i we have that

$$0 \cdot (i,k+i) = k \qquad \text{and} \qquad U_i(0)\cdot\{(i,i+k)\} = U_i(k),$$

where $U_i(0) \in \mathscr{B}_3(0)$ and $U_i(k) \in \mathscr{B}_3(k)$. Then the continuity of the semigroup operation in T implies that for every open neighbourhood $W(k)$ of k in T there exists $U_i(0) \in \mathscr{B}_3(0)$ such that

$$U_i(0)\cdot\{(i,i+k)\} = U_i(k) \subseteq W(k).$$

This implies that the bases of topologies τ and τ_3 at the point $k \in T$ coincide.

In the case when $k < 0$ the proof is similar. This completes the proof of our assertion. ◻

Theorem 4.7 implies the following:

Corollary 4.8. *Let T be a Hausdorff locally compact topological inverse semigroup. If T contains $\mathscr{C}_{\mathbb{Z}}$ as a dense subsemigroup such that $I = T \setminus \uparrow\mathscr{C}_{\mathbb{Z}} \neq \varnothing$ and $\uparrow\mathscr{C}_{\mathbb{Z}} = \mathscr{C}_{\mathbb{Z}}$, then T is topologically isomorphic to the topological inverse semigroup (S_3,τ_3) from Example 4.3.*

Theorem 4.9. *Let (T,τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. If $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that $\uparrow\mathscr{C}_{\mathbb{Z}} = T$ and the group of units of T is singleton, then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that (T,τ) is topologically isomorphic to the semigroup (S_1,τ_1) from Example 4.1.*

Proof. By the assumption of the theorem we have that $T \setminus \mathscr{C}_{\mathbb{Z}} = \{1_T\}$. Then Lemma 3.6$(i)$ implies that there exists a base $\mathscr{B}(1_T)$ of the topology τ at the unit 1_T such that $U(1_T) \subseteq E(\mathscr{C}_{\mathbb{Z}})$ for any $U(1_T) \in \mathscr{B}(1_T)$. Also statements (c) and (d) of Theorem 1.7 from [6, Vol. 1] imply that we can assume that $(n,n) \in U(1_T)$ if and only if n is a negative integer. Since by Corollary 3.2 every element of the semigroup $\mathscr{C}_{\mathbb{Z}}$ is an isolated point of T, without loss of generality we can assume that all elements of the base $\mathscr{B}(1_T)$ are closed-and-open subsets of T. Also, the local compactness of T implies that without loss of generality we can assume that the base $\mathscr{B}(1_T)$ consists of compact subsets, and Corollary 3.3.6 from [9] implies that the base $\mathscr{B}(1_T)$ is countable.

We suppose that $\mathscr{B}(1_T) = \{U_n(1_T) \mid n = 1,2,3,\ldots\}$. We put

$$W_1(1_T) = U_1(1_T) \qquad \text{and} \qquad W_i(1_T) = W_{i-1}(1_T) \cap U_i(1_T),$$

for all $i = 2,3,4,\ldots$. We observe that $\widetilde{\mathscr{B}}(1_T) = \{W_n(1_T) \mid n = 1,2,3,\ldots\}$ is a base of the topology τ at the unit 1_T of T such that $W_{n+1}(1_T) \subsetneqq W_n(1_T)$ for every positive integer n. Then the compactness of $U_i(1_T)$, $i = 1,2,3,\ldots$, and the discreteness of the space $\mathscr{C}_{\mathbb{Z}}$ imply that the family $\widetilde{\mathscr{B}}(1_T)$ consists of compact-and-open subsets of T. Let $\{m_i\}_{i\in\mathbb{N}}$ be a decreasing sequence of negative integers such that $\displaystyle\bigcup_{i=1}^{\infty}\{(m_i,m_i)\} = W_1(1_T) \setminus \{1_T\}$. We put $V_n = \{1_T\} \cup \{(m_i,m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant n\}$ for every positive

integer n. Since every element of the family $\widetilde{\mathscr{B}}(1_T)$ is a compact subset of T, Corollary 3.2 implies that the family

$$\overline{\mathscr{B}}(1_T) = \{V_n \mid n = 1, 2, 3, \ldots\}$$

is a base of the topology τ at 1_T of T and this completes the proof of our theorem. □

Theorems 4.7 and 4.9 imply the following:

Corollary 4.10. *Let (T, τ) be a Hausdorff locally compact topological inverse semigroup. If $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the group of units of T is singleton, then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that (T, τ) is topologically isomorphic either to the semigroup (S_1, τ_1) from Example 4.1 or to the semigroup (S_4, τ_4) from Example 4.4.*

Theorem 4.11. *Let (T, τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. Suppose that $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the following conditions hold:*

(*i*) *$\uparrow\mathscr{C}_{\mathbb{Z}} = T$;*
(*ii*) *the group of units $H(1_T)$ of T is non-singleton; and*
(*iii*) *there exists an integer j such that $K = \{1_T\} \cup \{(i, i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}$ is a compact subset of T.*

Then there exists a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i and (T, τ) is topologically isomorphic to the semigroup (S_2, τ_2) for $n = 1$ from Example 4.2.

Proof. As in the proof of Theorem 4.9 we construct a decreasing sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that the family

$$\mathscr{B}(1_T) = \{U_i(1_T) \mid i = 1, 2, 3, \ldots\}$$

determines a base of the topology τ at the point 1_T of T, where

$$U_j(1_T) = \{1_T\} \cup \{(m_i, m_i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}.$$

The compactness of the set K implies that we can construct a sequence of negative integers $\{m_i\}_{i\in\mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i.

Then for every element x of the group of units $H(1_T)$ left and right translations $\lambda_x\colon T \to T : s \mapsto x \cdot s$ and $\rho_x\colon T \to T : s \mapsto s \cdot x$ are homeomorphisms of the topological space T (see [6, Vol. 1, P. 19]), and hence the following families

$$\mathscr{B}_l(x) = \{x \cdot U_i(1_T) \mid U_i(1_T) \in \mathscr{B}(1_T)\}$$

and

$$\mathscr{B}_r(x) = \{U_i(1_T) \cdot x \mid U_i(1_T) \in \mathscr{B}(1_T)\}$$

are bases of the topology τ at the point 1_T of T. Also, we observe that the family

$$\mathscr{B}(x) = \{U \cap V \mid U \in \mathscr{B}_l(x) \text{ and } V \in \mathscr{B}_r(x)\}$$

is a base of the topology τ at the point 1_T of T.

Then Lemma 3.6 and Proposition 3.8 imply that the group of units $H(1_T)$ of T is topologically isomorphic to the discrete additive group of integers $\mathbb{Z}_+$. Let g be a generator of $\mathbb{Z}_+$. Then by Lemma 3.6(*iii*) there exist an open neighbourhood $U(g)$ of the point g in T and an integer k such that $a - b = k$ for all $(a, b) \in U(g) \cap \mathscr{C}_{\mathbb{Z}}$. Without loss of generality we can assume that g is a positive integer and $k < 0$. Then we have that

$$g \cdot U_i(1_T) = \{(m_i + k, m_i) \mid (m_i, m_i) \in U_i(1_T)\} \cup \{g\} \tag{3}$$

and

$$U_i(1_T) \cdot g = \{(m_i, m_i - k) \mid (m_i, m_i) \in U_i(1_T)\} \cup \{g\} \tag{4}$$

We shall show that equality (4) holds. Let be $(m_i, m_i) \in U_i(1_T)$. Then we get

$$((m_i, m_i) \cdot g) \cdot ((m_i, m_i) \cdot g)^{-1} = (m_i, m_i) \cdot g \cdot g^{-1} \cdot (m_i, m_i)^{-1} = (m_i, m_i) \cdot 1_T \cdot (m_i, m_i) = (m_i, m_i).$$

Since $(m_i, m_i) \cdot g \in \mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup we conclude that $(m_i, m_i) \cdot g = (m_i, a)$ for some integer a, and by Lemma 3.6(vi) we have that $(m_i, m_i) \cdot g = (m_i, m_i - k)$. This completes the proof of equality (4). The proof of equality (3) is similar. Then Lemma 3.6(vi), equalities (3) and (4) imply that T is topologically isomorphic to the semigroup (S_2, τ_2) for $n = 1$ from Example 4.2. This completes the proof of the theorem. $\square$

Theorems 4.7 and 4.11 imply the following:

Corollary 4.12. *Let (T, τ) be a Hausdorff locally compact topological inverse monoid with unit 1_T. Suppose that $\mathscr{C}_{\mathbb{Z}}$ is a dense subsemigroup of T such that the following conditions hold:*

(i) *the group of units $H(1_T)$ of T is non-singleton; and*

(ii) *there exists an integer j such that $K = \{1_T\} \cup \{(i,i) \in \mathscr{C}_{\mathbb{Z}} \mid i \geqslant j\}$ is a compact subset of T.*

Then there exists a decreasing sequence of negative integers $\{m_i\}_{i \in \mathbb{N}}$ such that $m_{i+1} = m_i - 1$ for every positive integer i and (T, τ) is topologically isomorphic either to the semigroup (S_2, τ_2) from Example 4.2 or to the semigroup (S_5, τ_5) from Example 4.5.

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